Physics 12c: Problem Set 1

Due: Thursday, April 11, 2019

1. **Sharply-peaked functions.** Stirling's approximation is

$$n! = (2\pi n)^{1/2} n^n e^{-n} \left(1 + \frac{1}{12n} + O(1/n^2)\right) \tag{1}$$

Derive it as follows

(a) Show that $n! = \Gamma(n+1)$, where

$$\Gamma(n) \equiv \int_0^\infty \frac{dx}{x} x^n e^{-x}. \tag{2}$$

First show that $\Gamma(1) = 1 = 0!$. Using integration by parts, show that $\Gamma(n)$ satisfies the identity

$$\Gamma(n+1) = n\Gamma(n). \tag{3}$$

Since this identity is also satisfied by $n!$, the claim follows.

(b) When n is large, the integrand for $\Gamma(n+1)$ is sharply peaked as a function of x, and gets most of its contribution from the region near the peak. Write the integrand as

$$x^n e^{-x} = \exp\left(n \log x - x\right). \tag{4}$$

Let x_0 be the point where the integrand achieves its maximum. Expand the quantity in the exponential around x_0,

$$n \log x - x = a_0 - a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + a_6 y^6 + \dots, \tag{5}$$

where $y = x - x_0$ and a_2 is positive. We have

$$\Gamma(n+1) = \int_{-x_0}^\infty dy \exp\left(a_0 - a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + a_6 y^6 \dots\right). \tag{6}$$

(c) Argue that in the large-n limit, the integrand gets most of its contribution from the region where $a_2 y^2$ is order 1. Show that the $a_k y^k$ for $k > 2$ are small in this region. Use this to justify expanding the exponential

$$\Gamma(n+1) = \int_{-x_0}^\infty dy e^{a_0 - a_2 y^2} \left(1 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \left(a_6 + \frac{a_3^2}{2}\right) y^6 + \dots\right). \tag{7}$$

* **Optional** The above expansion is good in the region where the gaussian $e^{-a_2 y^2}$ is peaked. However, it breaks down far away from that region, when y is large. This is not a problem as long as $I_k = \int_{-\infty}^{\infty} dy\, e^{-a_2 y^2} a_k y^k$ remains small (the e^{a_0} factor just multiplies the whole answer so it is unimportant for this analysis). However, argue that I_k can become large for sufficiently large k. Thus, we must truncate the series in the exponent (6) before expanding. Have a look at `https://en.wikipedia.org/wiki/Stirling's_approximation` for a plot of the relative error in the truncated Stirling series as a function of k. Note that for any fixed n, the error is initially decreasing as a function of k, and then rises again.

  The expansion (7) is an example of perturbation theory for Gaussian integrals. It is at the core of the Feynman diagram expansion for quantum field theory. It always produces series that must be truncated and are only correct up to "non-perturbative" corrections of the form e^{-n}. Such series are called "asymptotic series."

(d) Ingore the "…" and focus on the terms written in (7). Argue that when n is large, we can replace $\int_{-x_0}^{\infty} \to \int_{-\infty}^{\infty}$ up to small errors. Perform the Gaussian integrals to obtain Stirling's approximation.

2. **Central limit theorem.** Let $p(s)$ be a probability distribution, and let $\{s_1, \ldots, s_N\}$ be random variables drawn independently from that distribution. Let $S = \sum_{i=1}^{N} s_i$.

Theorem 0.1 (Central limit theorem)**.** *No matter what $p(s)$ is, the probability distribution $P(S)$ for S approaches a Gaussian as $N \to \infty$*

$$P(S) \to \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left(-\frac{(S - N\langle s\rangle)^2}{2N\sigma^2}\right), \quad \textit{as } N \to \infty \tag{8}$$

where $\sigma^2 = \langle s^2\rangle - \langle s\rangle^2$. Here, $\langle f\rangle$ denotes expectation value

$$\langle f\rangle = \int ds\, p(s) f(s). \tag{9}$$

In class, we studied a spin system where $s_i = \pm 1$. This corresponds to a Dirac-delta-function supported probability distribution $p(s)$:

$$p(s) = \frac{1}{2}(\delta(s-1) + \delta(s+1)). \tag{10}$$

We showed in this case that the distribution of the total spin excess $S = \sum_{i=1}^{N} s_i$ becomes a Gaussian with width $\sqrt{N}$ in the large-N limit. The central limit theorem shows that this is true even for general $p(s)$.

(a) Show that $\langle (s - \langle s\rangle)^2\rangle = \langle s^2\rangle - \langle s\rangle^2 = \sigma^2$.

(b) Define $S_n = \sum_{i=1}^n s_i$, so that $S = S_N$. Show the recursion relation

$$\langle S_n \rangle = \langle S_{n-1} \rangle + \langle s \rangle \tag{11}$$

Conclude that $\langle S \rangle = N \langle s \rangle$, which is consistent with (8).

(c) Using the same method, compute $\langle S^2 \rangle$ in terms of $\langle s \rangle, \langle s^2 \rangle$ and N. Write $\langle S^2 \rangle - \langle S \rangle^2$ in terms of σ^2. This is a generalization of our result about random walks from class.

(d) Show

$$\langle e^{itS} \rangle = \langle e^{its} \rangle^N. \tag{12}$$

(e) Expand (12) in a power series in t and recover your expressions from part (2c) in a different way. What are $\langle S^3 \rangle$ and $\langle S^4 \rangle$?

(f) Show that

$$P(S') = \langle \delta(S' - S) \rangle, \tag{13}$$

where $\delta(x)$ is the Dirac delta-function. (Hint: integrate both sides against a function of S'.) Show that $P(S')$ is the inverse Fourier transform of (12):

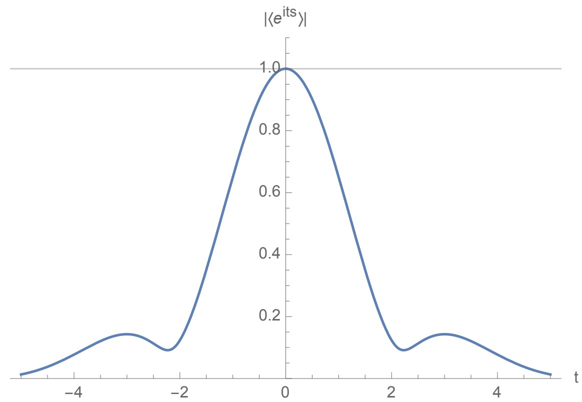
$$P(S') = \int \frac{dt}{2\pi} e^{-itS'} \langle e^{its} \rangle^N \tag{14}$$

(g) Let us write

$$\langle e^{its} \rangle = e^{a_0 + a_1 t - a_2 t^2 + \dots}. \tag{15}$$

Determine a_0, a_1, a_2 in terms of $\langle s \rangle, \langle s^2 \rangle$. In the large-$N$ limit, argue that (14) can be approximated by a Gaussian integral (i.e. one can ignore the "…" in (15).). Perform the Gaussian integral to recover (8).

Hint: Note that $\langle e^{its} \rangle$ is an average of phases. Thus, it has magnitude less than 1 unless all the phases e^{its} are precisely aligned. You may assume that $p(s)$ is sufficiently generic so that the phases only align when $t = 0$. (The phases could align at other values of t if $p(s)$ has delta-function support at integer multiples of some fixed value t_0, but let us assume this is not the case.) Thus, as a function of t, the magnitude $|\langle e^{its} \rangle|$ looks like this:



What happens when you raise this function to the N-th power? Use your method from problem (1c) to complete the argument that the "..." can be dropped.

* **Optional** Using the methods of problem 1, compute the first subleading correction to $P(S)$ in the large-N limit in terms of expectation values $\langle s^k \rangle$.

3. **Shannon entropy.** Suppose that we receive a message of length N consisting of a string of symbols a and b, for example

$$abbababababaabbbbbababababababbabb... \tag{16}$$

Suppose that a occurs with probability p and b occurs with probability $1-p$.

(a) When N is large, most messages will contain pN a's and $(1-p)N$ b's. Show that the number of such messages is approximately

$$2^{NS}, \tag{17}$$

where

$$S = -p \log_2 p - (1-p)\log_2(1-p). \tag{18}$$

S is called the **Shannon entropy** per letter. It is roughly the number of bits of information per letter.

(b) For what value of p is the number of bits per letter smallest? For what value of p is it largest?

(c) Suppose we have an alphabet with k letters $a_1, \ldots, a_k$, and the probability to observe a_i is p_i. Show that when N is large, the number of possible messages is approximately 2^{NS}, where

$$S = -\sum_{i=1}^{k} p_i \log_2 p_i. \tag{19}$$